Sistem linearnih jednačina

Sistem od m jednačina sa n nepoznatih zovemo sistem linearnih jednačina

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
\vdots \qquad\quad \vdots \qquad\ \ \vdots \quad & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m
\end{aligned}
$$

Sisteme linearnih jednačina možemo rješiti:

a) Gausovom metodom
b) Kramerovovom metodom (metoda determinanti)
c) Matričnom metodom
d) Kroneker-Kapelijevom metodom

1. Gausovom metodom rješiti sistem jednačina

$$
\begin{aligned}
x_1 + x_2 - 2x_3 + 4x_4 &= -1 \qquad (1)\\
3x_1 + 2x_2 - x_3 + 3x_4 &= 0 \qquad (2)\\
2x_1 - x_2 + 3x_3 - x_4 &= 9 \qquad (3)\\
5x_1 - 2x_2 + \underline{x_3} - 2x_4 &= 9 \qquad (4)
\end{aligned}
$$

Rj.

$(1)+2\cdot(4)$: $\cancel{11}x_1 - 3x_2 = 17$

$(2)+(4)$: $8x_1 + x_4 = 9$

$(3)-3\cdot(4)$: $-13x_1 + 5x_2 + 5x_4 = -18$

$3x_2 = 11x_1 - 17 \Rightarrow x_2 = \frac{1}{3}(11x_1 - 17)$

$x_4 = -8x_1 + 9$

$-13x_1 + 5x_2 + 5x_4 = -18$

$-13x_1 + \frac{5}{3}(11x_1 - 17) + 5(-8x_1 + 9) = -18$

$\underline{-13x_1} + \frac{55}{3}x_1 - \frac{85}{3} \underline{-40x_1} + 45 = -18$

$-53x_1 + \frac{55}{3}x_1 = -63 + \frac{85}{3} \quad /\cdot 3$

$-159x_1 + 55x_1 = -189 + 85$

$-104x_1 = -104$

$x_1 = 1$

$x_2 = \frac{1}{3}(11x_1 - 17) = \frac{1}{3}(11 - 17) = -2$

$x_4 = -8x_1 + 9 = 1$

$x_1 + x_2 - 2x_3 + 4x_4 = -1$

$-2x_3 = -1 + 2 - 4 - 1$

$-2x_3 = -4$

$x_3 = 2$

Rješenje sistema je $x_1 = 1,\ x_2 = -2,\ x_3 = 2,\ x_4 = 1$

2. Gausovom metodom riješiti sistem jednačina

$$
\begin{aligned}
2x_1 + 3x_2 - 5x_3 + x_4 - x_5 &= 0 \\
x_1 + 2x_2 + 3x_3 + 2x_4 + 2x_5 &= 3 \\
4x_1 + 7x_2 + x_3 + 5x_4 + 3x_5 &= 6 \\
5x_1 + 9x_2 + 4x_3 + 7x_4 + 5x_5 &= 9
\end{aligned}
$$

Riješiti sistem linearnih jednačina

$$2x_1 - 2x_2 + 2x_3 + 3x_4 = 1$$
$$-2x_1 + x_2 - x_3 - 4x_4 = 0$$
$$2x_1 - 3x_2 + 3x_3 + 2x_4 = 2$$
$$-x_2 + x_3 - x_4 = 1$$

Rj: Riješimo sistem Gausovom metodom:

$$\begin{array}{ll} 2x_1 - 2x_2 + 2x_3 + 3x_4 = 1 & (a) \\ -2x_1 + x_2 - x_3 - 4x_4 = 0 & (b) \\ 2x_1 - 3x_2 + 3x_3 + 2x_4 = 2 & (c) \\ -x_2 + x_3 - x_4 = 1 & (d) \end{array}$$

$$\begin{array}{ll} (a): & 2x_1 - 2x_2 + 2x_3 + 3x_4 = 1 \\ (b)+(a): & -x_2 + x_3 - x_4 = 1 \\ (c)-(a): & -x_2 + x_3 - x_4 = 1 \\ & -x_2 + x_3 - x_4 = 1 \end{array}$$

$$2x_1 - 2x_2 + 2x_3 + 3x_4 = 1$$
$$-x_2 + x_3 - x_4 = 1$$

Imamo dvije linearne jednačine sa četri nepoznate $\Rightarrow$

$\Rightarrow$ dvije promjenjive uzimamo proizvoljno npr. $x_3 = s$, $x_4 = t$

$$x_2 = s - t - 1$$

$$2x_1 = 1 + 2x_2 - 2x_3 - 3x_4$$
$$2x_1 = 1 + 2s - 2t - 2 - 2s - 3t$$
$$2x_1 = -5t - 1$$
$$x_1 = \frac{-5}{2}t - \frac{1}{2}$$

Rješenje sistema linearnih jednačina je

$$\left(\frac{-5}{2}t - \frac{1}{2}, s - t - 1, s, t\right)$$

Cramerovo pravilo (metoda determinanti)

Rješavamo sistem oblika $A\cdot x=b$ gdje je $A=[a_{ij}]_{n\times n}$, $x=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$

$b=\begin{bmatrix}b_1\\b_2\\\vdots\\b_n\end{bmatrix}$. D_k determinanta koja se dobije od D ($D=\det A$) kada se umjesto k-te kolone u D stave slobodni članovi $\begin{bmatrix}b_1\\b_2\\\vdots\\b_n\end{bmatrix}$.

a) za $D\neq0$ sistem ima jedinstveno rješenje $x=\frac{D_x}{D}$, $y=\frac{D_y}{D}$, $z=\frac{D_z}{D}$

b) za $D=0$ i ($D_x\neq0$ ili $D_y\neq0$ ili $D_z\neq0$) sistem nema nijedno rješenje

c) za $D=D_x=D_y=D_z$ ne možemo ništa zaključiti (sistem može imati ∞ mnogo rješenja ili nemati nijedno rješenje) (potrebna su dalja ispitivanja)

) Metodom determinanti riješiti sistem jednačina

$$2x-y-z=4$$
$$3x+4y-2z=11$$
$$3x-2y+4z=11.$$

Rj.

$$D=\begin{vmatrix}2&-1&-1\\3&4&-2\\3&-2&4\end{vmatrix}\underset{III_v+I_v\cdot4}{\overset{II_v+I_v\cdot(-2)}{=\!=}}\begin{vmatrix}2&-1&-1\\-1&6&0\\11&-6&0\end{vmatrix}=(-1)\begin{vmatrix}-1&6\\11&-6\end{vmatrix}=-(6-66)=60$$

$$D_x=\begin{vmatrix}4&-1&-1\\11&4&-2\\11&-2&4\end{vmatrix}\underset{III_v+I_v\cdot4}{\overset{II_v-I_v\cdot2}{=\!=}}\begin{vmatrix}4&-1&-1\\3&6&0\\27&-6&0\end{vmatrix}=(-1)\begin{vmatrix}3&6\\27&-6\end{vmatrix}=-(-18-162)=180$$

$$D_y=\begin{vmatrix}2&4&-1\\3&11&-2\\3&11&4\end{vmatrix}\underset{II_k+III_k\cdot4}{\overset{I_k+III_k\cdot2}{=\!=}}\begin{vmatrix}0&0&-1\\-1&3&-2\\11&27&4\end{vmatrix}=(-1)\begin{vmatrix}-1&3\\11&27\end{vmatrix}=-(-27-33)=60$$

$$D_z=\begin{vmatrix}2&-1&4\\3&4&11\\3&-2&11\end{vmatrix}\underset{III_v-I_v\cdot2}{\overset{II_v+I_v\cdot4}{=\!=}}\begin{vmatrix}2&-1&4\\11&0&27\\-1&0&3\end{vmatrix}=\begin{vmatrix}11&27\\-1&3\end{vmatrix}=3\begin{vmatrix}11&9\\-1&1\end{vmatrix}=3(11+9)=60$$

$$x=\frac{D_x}{D}=\frac{180}{60}=3;\quad y=\frac{D_y}{D}=\frac{60}{60}=1;\quad z=\frac{D_z}{D}=\frac{60}{60}=1$$

Rješenje sistema je $x=3$, $y=1$ i $z=1$

)^v Metodom determinanti riješiti sistem jednačina:

$$2x+4y-5z=-5$$
$$-x-y+z=0$$
$$2x+y-z=1.$$

Rj. $x=1$, $y=2$, $z=3$

) Riješiti sistem jednačina i diskutovati rješenja u zavisnosti od parametra λ:

$$(\lambda-2)x - 3y + 2z = 1$$
$$3x - 3y + (\lambda-3)z = 1$$
$$x - y + 2z = -1 .$$

Rj:

$$D=\begin{vmatrix} \lambda-2 & -3 & 2 \\ 3 & -3 & \lambda-3 \\ 1 & -1 & 2 \end{vmatrix} \underset{III_k+II_k\cdot 2}{\overset{I_k+II_k}{=\!=}} \begin{vmatrix} \lambda-5 & -3 & -4 \\ 0 & -3 & \lambda-9 \\ 0 & -1 & 0 \end{vmatrix} = (\lambda-5)\begin{vmatrix} -3 & \lambda-9 \\ -1 & 0 \end{vmatrix} = (\lambda-5)(\lambda-9)$$

$$D_x=\begin{vmatrix} 1 & -3 & 2 \\ 1 & -3 & \lambda-3 \\ -1 & -1 & 2 \end{vmatrix} \underset{II_v+III_v}{\overset{I_v+III_v}{=\!=}} \begin{vmatrix} 0 & -4 & 4 \\ 0 & -4 & \lambda-1 \\ -1 & -1 & 2 \end{vmatrix} = (-1)\begin{vmatrix} -4 & 4 \\ -4 & \lambda-1 \end{vmatrix} = (-1)(-4)\begin{vmatrix} 1 & 4 \\ 1 & \lambda-1 \end{vmatrix} = 4(\lambda-5)$$

$$D_y=\begin{vmatrix} \lambda-2 & 1 & 2 \\ 3 & 1 & \lambda-3 \\ 1 & -1 & 2 \end{vmatrix} \underset{II_v+III_v}{\overset{I_v+III_v}{=\!=}} \begin{vmatrix} \lambda-1 & 0 & 4 \\ 4 & 0 & \lambda-1 \\ 1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} \lambda-1 & 4 \\ 4 & \lambda-1 \end{vmatrix} = (\lambda-1)^2-4 = (\lambda-1-4)(\lambda-1+4) = (\lambda-5)(\lambda+3)$$

$$D_z=\begin{vmatrix} \lambda-2 & -3 & 1 \\ 3 & -3 & 1 \\ 1 & -1 & -1 \end{vmatrix} \overset{I_k+II_k}{=\!=} \begin{vmatrix} \lambda-5 & -3 & 1 \\ 0 & -3 & 1 \\ 0 & -1 & -1 \end{vmatrix} = (\lambda-5)\begin{vmatrix} -3 & 1 \\ -1 & -1 \end{vmatrix} = 4(\lambda-5)$$

Diskusija

1° $\lambda \neq 5$ i $\lambda \neq 9$ $(D\neq 0)$ Sistem ima jedinstveno rješenje

$$x=\frac{D_x}{D}=\frac{4(\lambda-5)}{(\lambda-5)(\lambda-9)}=\frac{4}{\lambda-9}, \quad y=\frac{D_y}{D}=\frac{\lambda+3}{\lambda-9}, \quad z=\frac{D_z}{D}=\frac{4}{\lambda-9}$$

2° $\lambda=9$

$D=0,\ D_x\neq 0 \Rightarrow$ sistem nema rješenja

3° $\lambda=5 \Rightarrow D=D_x=D_y=D_z=0$ na osnovu Cramerovog pravila ne možemo ništa zaključiti. A trebno je uraditi sistem na drugi način.

za $\lambda=5$ sistem postaje

$$3x-3y+2z=1 \quad (1)$$
$$3x-3y+2z=1 \quad (2)$$
$$x-y+2z=-1 \quad (3)$$

$(1)=(2)$

$(2)-(3):\ 2x-2y=2$

$x=y+1$

$x-y+2z=-1$

$y+1-y+2z=-1$

$2z=-2$

$z=-1$

sistem ima beskonačno mnogo rješenja koji su oblika $(t+1,\ t,\ -1),\ t\in\mathbb{R}$

) Riješiti sistem jednačina i diskutovati rješenja u zavisnosti od parametra λ:

$$(\lambda+4)x + y + z = 2$$
$$x + y + z = \lambda+5$$
$$3x + 3y + (\lambda+7)z = 3$$

Rj: $D=(\lambda+4)(\lambda+3)$

$D_x=-(\lambda+4)(\lambda+3)$

$D_y=(\lambda+3)(\lambda+4)(\lambda+8)$

$D_z=-3(\lambda+3)(\lambda+4)$

$t\in\mathbb{R}$

$(t, 5-t, -3)$

$(-1, 2-s, s)$

$s\in\mathbb{R}$

(#) Riješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra λ

$$x+y+z=4$$
$$x+\lambda y+z=3$$
$$x+2\lambda y+z=4 \quad .$$

Rj. Sistem rješavamo Cramerovom metodom

$$D=\begin{vmatrix}1&1&1\\1&\lambda&1\\1&2\lambda&1\end{vmatrix} \overset{II_v-III_v}{=\!=\!=} \begin{vmatrix}1&1&1\\0&-\lambda&0\\1&2\lambda&1\end{vmatrix} = -\lambda\begin{vmatrix}1&1\\1&1\end{vmatrix}=0$$

$$D_x=\begin{vmatrix}4&1&1\\3&\lambda&1\\4&2\lambda&1\end{vmatrix} \underset{III_v-II_v}{\overset{I_v-II_v}{=\!=\!=}} \begin{vmatrix}1&1-\lambda&0\\3&\lambda&1\\1&\lambda&0\end{vmatrix} = -\begin{vmatrix}1&1-\lambda\\1&\lambda\end{vmatrix} = -(\lambda-(1-\lambda)) = 1-\lambda-\lambda = 1-2\lambda$$

$$D_y=\begin{vmatrix}1&4&1\\1&3&1\\1&4&1\end{vmatrix} \overset{III_k-I_k}{=\!=\!=} \begin{vmatrix}1&4&0\\1&3&0\\1&4&0\end{vmatrix} = 0$$

$$D_z=\begin{vmatrix}1&1&4\\1&\lambda&3\\1&2\lambda&4\end{vmatrix} \underset{III_v-II_v}{\overset{I_v-II_v}{=\!=\!=}} \begin{vmatrix}0&1-\lambda&1\\1&\lambda&3\\0&\lambda&1\end{vmatrix} = -\begin{vmatrix}1-\lambda&1\\\lambda&1\end{vmatrix} = -(1-\lambda-\lambda) = 2\lambda-1$$

Kako je $D=0$ to sistem može da ima beskonačno mnogo rješenja ili da nema rješenja.

$1°$ $\lambda=\frac{1}{2}$

$D=0,\ D_x=0,\ D_y=0,\ D_z=0$

Sistem ćemo riješiti Gauss-ovom metodom

$$x+y+z=4$$
$$x+\tfrac{1}{2}y+z=3 \quad /\cdot 2$$
$$x+y+z=4$$

$$x+y+z=4 \quad (1)$$
$$2x+y+2z=6 \quad (2)$$

$(2)-(1):\ \ x+z=2$

$x=2-z$

$$x+y+z=4$$
$$2-z+y+z=4$$
$$y=2$$

Za $\lambda=\frac{1}{2}$ sistem ima ∞ mnogo rješenja koja su oblika $(2-t, 2, t)$ gdje je $t\in\mathbb{R}$.

$2°$ $\lambda\neq\frac{1}{2}$

$D=0,\ D_x\neq 0 \Rightarrow$ sistem za $\lambda\neq\frac{1}{2}$ nema rješenja

(#) Odrediti vrijednost parametra k tako da sistem

$$8z - 3x - 6y = kx$$
$$2x + y + 4z = ky$$
$$4x + 3y + z = kz$$

ima beskonačno mnogo rješenja. Zatim naći ta rješenja za najveću dobijenu vrijednost parametra k.

Rj: Nepoznate sa desne strane prebacimo na lijevu i grupirajmo vrijednosti uz x, y i z.

$$(-3-k)x - 6y + 8z = 0$$
$$2x + (1-k)y + 4z = 0$$
$$4x + 3y + (1-k)z = 0$$

$$\begin{vmatrix} -3-k & -6 & 8 \\ 2 & 1-k & 4 \\ 4 & 3 & 1-k \end{vmatrix} = 0$$

Ovo je homogeni sistem linearnih jednačina. Trivijalno rješenje je $(0,0,0)$. Sistem ima beskonačno mnogo rješenja ako je $D=0$.

$I_k + III_k$: $$\begin{vmatrix} 5-k & -6 & 8 \\ 6 & 1-k & 4 \\ 5-k & 3 & 1-k \end{vmatrix} = 0$$

$I_v - III_v$: $$\begin{vmatrix} 0 & -9 & 7+k \\ 6 & 1-k & 4 \\ 5-k & 3 & 1-k \end{vmatrix} = 0$$

$-9+9k \qquad -21-3k \qquad 7-6k-k^2 \qquad 7-7k+k-k^2$

$$6\left[(-9)(1-k) - 3(7+k)\right] + (5-k)\left[(-9)\cdot 4 - (7+k)(1-k)\right] = 0$$

$$(-6)(6k - 30) + (5-k)(\overset{-43}{\overbrace{-36 - 7}} + 6k + k^2) = 0$$

$$-36k + 180 + (-215) + 30k + 5k^2 + 43k - 6k^2 - k^3 = 0$$

$$-k^3 - k^2 + 37k - 35 = 0 \quad /\cdot(-1)$$

$$k^3 + k^2 - 37k + 35 = 0$$
$$k^3 - k^2 + 2k^2 - 2k - 35k + 35 = 0$$
$$k^2(1-k) + 2k(k-1) - 35(k-1) = 0$$

$$(k-1)(k^2 + 2k - 35) = 0$$
$$(k-1)(k+7)(k-5) = 0$$
$$k_1 = 1,\ k_2 = -7,\ k_3 = 5$$

Za $k=5$ imamo:

$$8x + 6y - 8z = 0 \quad \ldots(1)$$
$$2x - 4y + 4z = 0 \quad \ldots(2)$$
$$4x + 3y - 4z = 0 \quad \ldots(3)$$

$(1) \equiv (3)$ jer se (3) dobija dijeljenjem (1) sa 2.

$(2)+(3)$: $6x - y = 0$
$\Rightarrow y = 6x$

$(2) \Rightarrow 2x - 24x + 4z = 0$
$\therefore 4z = 22x$
$z = \frac{11x}{2}$

Za $k=5$ sistem ima rješenje $(t, 6t, \frac{11t}{2})$ gdje je $t \in \mathbb{R}$ proizvoljno.

(#) Riješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra λ:

$$\begin{aligned} x - y - \lambda z &= 1 \\ (\lambda+1)y + (\lambda-1)z &= 0 \\ (\lambda+1)x - (\lambda+1)z &= 1 \end{aligned}$$

Rj.

$$D = \begin{vmatrix} 1 & -1 & -\lambda \\ 0 & \lambda+1 & \lambda-1 \\ \lambda+1 & 0 & -(\lambda+1) \end{vmatrix} \overset{III_k + I_k}{=\!=} \begin{vmatrix} 1 & -1 & 1-\lambda \\ 0 & \lambda+1 & \lambda-1 \\ \lambda+1 & 0 & 0 \end{vmatrix} = (\lambda+1)\begin{vmatrix} -1 & -(\lambda-1) \\ \lambda+1 & \lambda-1 \end{vmatrix} =$$

$$= (\lambda+1)(\lambda-1)\begin{vmatrix} -1 & -1 \\ \lambda+1 & 1 \end{vmatrix} = \lambda(\lambda-1)(\lambda+1)$$

$$D_x = \begin{vmatrix} 1 & -1 & -\lambda \\ 0 & \lambda+1 & \lambda-1 \\ 1 & 0 & -(\lambda+1) \end{vmatrix} \overset{III_v - I_v}{=\!=} \begin{vmatrix} 1 & -1 & -\lambda \\ 0 & \lambda+1 & \lambda-1 \\ 0 & 1 & -1 \end{vmatrix} = \begin{vmatrix} \lambda+1 & \lambda-1 \\ 1 & -1 \end{vmatrix} = \lambda-1-\lambda+1$$

$-1+\lambda+1$

$$= -2\lambda$$

$$D_y = \begin{vmatrix} 1 & 1 & -\lambda \\ 0 & 0 & \lambda-1 \\ \lambda+1 & 1 & -(\lambda+1) \end{vmatrix} = -(\lambda-1)\begin{vmatrix} 1 & 1 \\ \lambda+1 & 1 \end{vmatrix} = -(\lambda-1)(1-\lambda-1) = \lambda(\lambda-1)$$

$$D_z = \begin{vmatrix} 1 & -1 & 1 \\ 0 & \lambda+1 & 0 \\ \lambda+1 & 0 & 1 \end{vmatrix} = (\lambda+1)\begin{vmatrix} 1 & 1 \\ \lambda+1 & 1 \end{vmatrix} = -\lambda(\lambda+1)$$

$D=0$ akko $\lambda=0$ ili $\lambda=1$ ili $\lambda=-1$

Diskusija

1° $\lambda \neq 0$ i $\lambda \neq 1$ i $\lambda \neq -1$ sistem ima jedinstveno rješenje

$$x = \frac{D_x}{D} = \frac{-2\lambda}{\lambda(\lambda-1)(\lambda+1)} = \frac{-2}{(\lambda-1)(\lambda+1)}, \quad y = \frac{D_y}{D} = \frac{1}{\lambda+1}, \quad z = \frac{D_z}{D} = \frac{-1}{\lambda+1}$$

2° $\lambda=1$, $D=0$, $D_x \neq 0$ $\Rightarrow$ sistem nema rješenja

3° $\lambda=-1$, $D=0$, $D_x \neq 0$ $\Rightarrow$ sistem nema rješenja

4° $\lambda=0$, $D=D_x=D_y=D_z=0$ iz ovoga ne možemo ništa zaključiti

Za $\lambda=0$ sistem postaje

$$\begin{aligned} x - y \quad &= 1 \quad (1) \\ y - z &= 0 \quad (2) \\ x \quad - z &= 1 \quad (3) \end{aligned}$$

(1): $x-y=1$

(2)−(3): $-x+y=-1$

$x=y+1$

$x-z=1$

$-z=-(y+1)+1$

$-z=-y$

$z=y$

Sistem ima ∞ mnogo rješenja $(t+1, t, t)$, $t \in \mathbb{R}$

#) Riješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra a :

$$x+y-z=0$$
$$x-y+az=1$$
$$-x-3y+(a+2)z=a^2 .$$

Rj.

$$D=\begin{vmatrix}1&1&-1\\1&-1&a\\-1&-3&a+2\end{vmatrix}\underset{II_k+III_k}{\overset{I_k+III_k}{=\!=}}\begin{vmatrix}0&0&-1\\a+1&a-1&a\\a+1&a-1&a+2\end{vmatrix}=(-1)\begin{vmatrix}a+1&a-1\\a+1&a-1\end{vmatrix}=0$$

$$D_x=\begin{vmatrix}0&1&-1\\1&-1&a\\a^2&-3&a+2\end{vmatrix}\overset{II_k+III_k}{=\!=}\begin{vmatrix}0&0&-1\\1&a-1&a\\a^2&a-1&a+2\end{vmatrix}=(-1)\begin{vmatrix}1&a-1\\a^2&a-1\end{vmatrix}=(-1)(a-1)\begin{vmatrix}1&1\\a^2&1\end{vmatrix}$$
$$=(-1)(a-1)(1-a^2)=(a-1)(a^2-1)$$
$$=(a-1)^2(a+1)$$

$$D_y=\begin{vmatrix}1&0&-1\\1&1&a\\-1&a^2&a+2\end{vmatrix}\overset{I_k+III_k}{=\!=}\begin{vmatrix}0&0&-1\\a+1&1&a\\a+1&a^2&a+2\end{vmatrix}=(-1)\begin{vmatrix}a+1&1\\a+1&a^2\end{vmatrix}=(-1)(a+1)\begin{vmatrix}1&1\\1&a^2\end{vmatrix}=(-1)(a+1)(a^2-1)$$
$$=(-1)(a-1)(a+1)^2$$

$$D_z=\begin{vmatrix}1&1&0\\1&-1&1\\-1&-3&a^2\end{vmatrix}\overset{I_k-II_k}{=\!=}\begin{vmatrix}0&1&0\\2&-1&1\\2&-3&a^2\end{vmatrix}=(-1)\begin{vmatrix}2&1\\2&a^2\end{vmatrix}=(-1)(2a^2-2)=(-2)(a+1)(a-1)$$

Diskusija

$D=0 \quad \forall a\in\mathbb{R}$

1° $a\neq 1$; $a\neq -1$

$D=0$; $D_x\neq 0$ sistem nema rješenja

2° $a=1$

$D=D_x=D_y=D_z=0$, sistem postaje

$$x+y-z=0 \quad (1)$$
$$x-y+z=1 \quad (2)$$
$$-x-3y+3z=1 \quad (3)$$

$(1)+(3)$: $-2y+2z=1$

$(2)+(3)$: $-4y+4z=2$

$2z=2y+1$

$z=y+\frac{1}{2}$

$x=z-y$

$x=\frac{1}{2}$

Sistem ima ∞ mnogo rješenja oblika $\left(\frac{1}{2}, t, t+\frac{1}{2}\right)$ gdje je $t\in\mathbb{R}$.

3° $a=-1$

$D=D_x=D_y=D_z=0$, sistem postaje

$$x+y-z=0 \quad I$$
$$x-y-z=1 \quad II$$
$$-x-3y+z=1 \quad III$$

$(I)+(III)$: $-2y=1$

$(II)+(III)$: $-4y=2$

$y=-\frac{1}{2}$

$(I)+(II)$: $2x-2z=1$

$(III)-3\cdot(II)$: $-4x+4z=2$

$2x=2z+1$

$x=z+\frac{1}{2}$

Sistem ima ∞ mnogo rješenja oblika $\left(t+\frac{1}{2}, -\frac{1}{2}, t\right)$, $t\in\mathbb{Z}$

(#) Diskutovati rješenja sistema u zavisnosti od parametra λ:

$$
\begin{aligned}
2x - \lambda y + 2z &= 1\\
x + y + 2z &= 0\\
-x + (-\lambda-3)y - 4z &= \lambda .
\end{aligned}
$$

Rj. Sistem ćemo riješiti Cranerovim pravilima.

$$
D = \begin{vmatrix} 2 & -\lambda & 2 \\ 1 & 1 & 2 \\ -1 & -\lambda-3 & -4 \end{vmatrix} \overset{I_k - II_k}{\underset{III_k - II_k\cdot 2}{=}} \begin{vmatrix} 2+\lambda & -\lambda & 2\lambda+2 \\ 0 & 1 & 0 \\ \lambda+2 & -\lambda-3 & 2\lambda+2 \end{vmatrix} = \begin{vmatrix} \lambda+2 & 2\lambda+2 \\ \lambda+2 & 2\lambda+2 \end{vmatrix} = (\lambda+2)\begin{vmatrix} 1 & 2\lambda+2 \\ 1 & 2\lambda+2 \end{vmatrix} = 0
$$

$$
D_x = \begin{vmatrix} 1 & -\lambda & 2 \\ 0 & 1 & 2 \\ \lambda & -\lambda-3 & -4 \end{vmatrix} \overset{III_k - II_k\cdot 2}{=} \begin{vmatrix} 1 & -\lambda & 2\lambda+2 \\ 0 & 1 & 0 \\ \lambda & -\lambda-3 & 2\lambda+2 \end{vmatrix} = \begin{vmatrix} 1 & 2\lambda+2 \\ \lambda & 2\lambda+2 \end{vmatrix} = (2\lambda+2)\begin{vmatrix} 1 & 1 \\ \lambda & 1 \end{vmatrix}
$$

$$
= (2\lambda+2)(1-\lambda) = 2(\lambda+1)(1-\lambda)
$$

$$
D_y = \begin{vmatrix} 2 & 1 & 2 \\ 1 & 0 & 2 \\ -1 & \lambda & -4 \end{vmatrix} \overset{III_k - I_k\cdot 2}{=} \begin{vmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ -1 & \lambda & -2 \end{vmatrix} = (-1)\begin{vmatrix} 1 & -2 \\ \lambda & -2 \end{vmatrix} = (-1)(-2)\begin{vmatrix} 1 & 1 \\ \lambda & 1 \end{vmatrix} = 2(1-\lambda)
$$

$$
D_z = \begin{vmatrix} 2 & -\lambda & 1 \\ 1 & 1 & 0 \\ -1 & -\lambda-3 & \lambda \end{vmatrix} \overset{I_k - II_k}{=} \begin{vmatrix} 2+\lambda & -\lambda & 1 \\ 0 & 1 & 0 \\ \lambda+2 & -\lambda-3 & \lambda \end{vmatrix} = \begin{vmatrix} \lambda+2 & 1 \\ \lambda+2 & \lambda \end{vmatrix} = (\lambda+2)\begin{vmatrix} 1 & 1 \\ 1 & \lambda \end{vmatrix} =
$$

$$
(\lambda+2)(\lambda-1)
$$

Diskusija:

$D=0$, $D_x = 2(1+\lambda)(1-\lambda)$, $D_y = 2(1-\lambda)$, $D_z = (\lambda+2)(\lambda-1)$

1° $\lambda \neq -1$; $\lambda \neq 1$; $\lambda \neq -2$

imamo $D=0$; $D_x \neq 0$ sistem nema rješenja

2° $\lambda = -2$ imamo $D=0$; $D_x \neq 0$ sistem nema rješenja

3° $\lambda = -1$ imamo $D=0$, $D_x = 0$, $D_y \neq 0$ sistem nema rješenja

4° $\lambda = 1$ imamo $D = D_x = D_y = D_z = 0$ sistem je potrebno ispitati na drugi način.

Za $\lambda = 1$ sistem postaje

$$
\begin{aligned}
2x - y + 2z &= 1 \quad /\cdot 4\\
x + y + 2z &= 0 \quad /\cdot 4\\
-x - 4y - 4z &= 1
\end{aligned}
$$

$$
\begin{aligned}
8x - 4y + 8z &= 4 \quad (1)\\
4x + 4y + 8z &= 0 \quad (2)\\
-x - 4y - 4z &= 1 \quad (3)
\end{aligned}
$$

$(1)+(2)$: $12x + 16z = 4$

$(3)+(2)$: $3x + 4z = 1$

$3x = 1 - 4z$

$x = \dfrac{1-4z}{3}$

$y = -x - 2z$

$y = \dfrac{4z-1}{3} - \dfrac{6z}{3}$

Sistem ima ∞ mnogo rješenja, oblika

$$
\left(\frac{1-4t}{3}, \frac{-2t-1}{3}, t\right)
$$

$t \in \mathbb{R}$

Riješiti sistem jednačina i diskutovati rješenja u zavisnosti od parametra

$$x+y+bz=1-b$$
$$x-by-z=2$$
$$bx-y+z=2b.$$

Rj. Rješavamo sistem Cramerovom metodom

$$D=\begin{vmatrix}1&1&b\\1&-b&-1\\b&-1&1\end{vmatrix}\xlongequal{I_k+III_k}\begin{vmatrix}b+1&1&b\\0&-b&-1\\b+1&-1&1\end{vmatrix}=(b+1)\begin{vmatrix}1&1&b\\0&-b&-1\\1&-1&1\end{vmatrix}\xlongequal{I_v-III_v}$$

$$=(b+1)\begin{vmatrix}0&2&b-1\\0&-b&-1\\1&-1&1\end{vmatrix}=(b+1)\begin{vmatrix}2&b-1\\-b&-1\end{vmatrix}=(b+1)\Big[\overbrace{-2+(b^2-b)}^{b^2-b-2}\Big]=$$

$$=(b+1)(b+1)(b-2)$$

$$D_x=\begin{vmatrix}1-b&1&b\\2&-b&-1\\2b&-1&1\end{vmatrix}\xlongequal{I_v+III_v}\begin{vmatrix}b+1&0&b+1\\2&-b&-1\\2b&-1&1\end{vmatrix}=(b+1)\begin{vmatrix}1&0&1\\2&-b&-1\\2b&-1&1\end{vmatrix}=$$

$$\xlongequal{I_k-III_k}(b+1)\begin{vmatrix}0&0&1\\3&-b&-1\\2b-1&-1&1\end{vmatrix}=(b+1)\begin{vmatrix}3&-b\\2b-1&-1\end{vmatrix}=(b+1)(\overbrace{-3+2b^2-b}^{2b^2-b-3})=$$

$$D=1+24=25 \qquad b_{1,2}=\frac{1\pm5}{4}$$

$$=(b+1)\cdot2\left(b-\frac{3}{2}\right)(b+1)$$
$$=(b+1)(2b-3)(b+1)$$

$$D_y=\begin{vmatrix}1&1-b&b\\1&2&-1\\b&2b&1\end{vmatrix}\xlongequal{I_k+III_k}\begin{vmatrix}b+1&1-b&b\\0&2&-1\\b+1&2b&1\end{vmatrix}=(b+1)\begin{vmatrix}1&1-b&b\\0&2&-1\\1&2b&1\end{vmatrix}\xlongequal{III_v-I_v}$$

$$=(b+1)\begin{vmatrix}1&1-b&b\\0&2&-1\\0&3b-1&1-b\end{vmatrix}=(b+1)\begin{vmatrix}2&-1\\3b-1&1-b\end{vmatrix}=(b+1)(2-2b+3b-1)=$$
$$=(b+1)(b+1)$$

$$D_z=\begin{vmatrix}1&1&1-b\\1&-b&2\\b&-1&2b\end{vmatrix}\xlongequal{I_v+III_v}\begin{vmatrix}b+1&0&b+1\\1&-b&2\\b&-1&2b\end{vmatrix}=(b+1)\begin{vmatrix}1&0&1\\1&-b&2\\b&-1&2b\end{vmatrix}\xlongequal{I_k-III_k}$$

$$=(b+1)\begin{vmatrix}0&0&1\\-1&-b&2\\-b&-1&2b\end{vmatrix}=(b+1)\begin{vmatrix}-1&-b\\-b&-1\end{vmatrix}=(b+1)(1-b^2)=-(b+1)(b^2-1)$$
$$=-(b+1)(b-1)(b+1)$$

Diskusija: a) $D\neq0$ tj. $b\neq-1$; $b\neq2$

sistem ima jedinstveno rješenje $x=\dfrac{D_x}{D}=\dfrac{(2b-3)(b+1)^2}{(b+1)^2(b-2)}=\dfrac{2b-3}{b-2}$

$$y=\frac{D_y}{D}=\frac{(b+1)^2}{(b+1)^2(b-2)}=\frac{1}{b-2}\quad;\quad z=\frac{D_z}{D}=\frac{-(b-1)(b+1)^2}{(b-2)(b+1)^2}=-\frac{b-1}{b-2}$$

b) $b=-1$ $\Rightarrow$ $D=D_x=D_y=D_z=0$ sistem trebamo rješiti na drugi način

Za $b=-1$ sistem postaje

$$\begin{array}{r} x+y-z=2 \\ x+y-z=2 \\ -x-y+z=-2 \quad /\cdot(-1) \\ \hline \end{array}$$

Sve tri jednačine su iste $\Rightarrow$ Sistem ima ∞ mnogo rješenja. Ako uzmemo $x=t$, $y=s$ Rješenja sistema su $(t, s, t+s-2)$

dvije promjenjive uzimamo proizvoljno

c) $b=2$ $\Rightarrow$ $D=0$, $D_x=9\neq 0$ $\Rightarrow$

Sistem za $b=2$ nema rješenja

Kroneker-Kapelijeva metoda

Neka je dat sistem linearnih jednačina $Ax=b$, gdje su

$$A=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n}, \quad x=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad i \quad b=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Matricu $\overline{A}=[A \,\vdots\, b]$ zovemo proširena matrica.

<u>Teorema</u> (Kroneker-Kapeli):

Sistem ima jedinstveno rješenje ako i samo ako je $\operatorname{rang} A = \operatorname{rang} \overline{A} = n$ (n broj nepoznatih).

Ako je $\operatorname{rang} A = \operatorname{rang} \overline{A} < n$ tada sistem ima ∞ mnogo rješenja. ($n - \operatorname{rang} A$ nepoznatih uzima se proizvoljno)

Ako je $\operatorname{rang} A < \operatorname{rang} \overline{A}$ tada sistem nema rješenja.

1. Kroneker-Kapelijevom metodom rješiti sistem jednačina

$$\begin{aligned} 2x + 4y - 5z &= -5 \\ -x - y + z &= 0 \\ 2x + y - z &= 1. \end{aligned}$$

Rj.

$$\overline{A}=[A \,\vdots\, b] = \left[\begin{array}{ccc|c} 2 & 4 & -5 & -5 \\ -1 & -1 & 1 & 0 \\ 2 & 1 & -1 & 1 \end{array}\right] \underset{I_v \leftrightarrow II_v}{\sim} \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 2 & 4 & -5 & -5 \\ 2 & 1 & -1 & 1 \end{array}\right] \underset{III_v + I_v \cdot 2}{\overset{II_v + I_v \cdot 2}{\sim}} \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & 2 & -3 & -5 \\ 0 & -1 & 1 & 1 \end{array}\right]$$

$$\underset{II_v \leftrightarrow III_v}{\sim} \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 2 & -3 & -5 \end{array}\right] \underset{III_v + II_v \cdot 2}{\sim} \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & -3 \end{array}\right]$$

$\operatorname{rang} A = \operatorname{rang} \overline{A} = 3$

sistem ima jedinstveno rješenje

$$\begin{aligned} -x - y + z &= 0 \\ -y + z &= 1 \\ -z &= -3 \end{aligned}$$

$z = 3$

$-x - y = -3$

$-y = -2$

$y = 2$

$-x - 2 = -3$

$x = 1$

Rješenje sistema je uređena trojka $(1,2,3)$.

2. Kroneker-Kapelijevom metodom rješiti sistem jednačina

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 3x_1 + x_2 - x_3 &= 3 \\ 2x_1 + x_2 &= 2. \end{aligned}$$

Rj. $\bar{A} = [A \mid b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & 3 \\ 2 & 1 & 0 & 2 \end{array}\right] \underset{III_v - I_v \cdot 2}{\overset{II_v - I_v \cdot 3}{\sim}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & -4 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \overset{II_v \leftrightarrow III_v}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & -2 & -4 & 0 \end{array}\right]$

$\overset{III_v - II_v \cdot 2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\text{rang}\, A = \text{rang}\, \bar{A} = 2 < 3$

sistem ima ∞ mnogo rješenja

$3-2$ nepoznatih uzimamo proizvoljno

$x_3 = t$

$-x_2 - 2x_3 = 0$

$x_1 + x_2 + x_3 = 1$

$-x_2 - 2t = 0$

$x_2 = -2t$

$x_1 - 2t + t = 1$

$x_1 = t + 1$

Sistem ima beskonačno mnogo rješenja oblika $(t+1, -2t, t)$ gdje je $t \in \mathbb{R}$.

3. Kroneker-Kapelijevom metodom rješiti sistem jednačina

$$\begin{aligned} x + 2y + 3z &= 1 \\ 2x + 4y + 6z &= 2 \\ 3x + 6y + 9z &= 5. \end{aligned}$$

Rj. $\bar{A} = [A \mid b] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 3 & 6 & 9 & 5 \end{array}\right] \underset{III_v - I_v \cdot 3}{\overset{II_v - I_v \cdot 2}{\sim}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right]$

$\text{rang}\, A = 1$, $\text{rang}\, \bar{A} = 2$, $\text{rang}\, A < \text{rang}\, \bar{A}$

sistem nema rješenja.

4. Kroneker-Kapelijevom metodom diskutovati rješenja sistema za razne vrijednosti parametra λ

$$\begin{aligned} \lambda x + y + z &= 1 \\ x + \lambda y + z &= 2 \\ x + y + \lambda z &= -3 \end{aligned}$$

Rj. za $\lambda \in (-\infty, -2) \cup (-2, 1) \cup (1, +\infty)$ sistem ima jedinstveno rješenje $\left(\frac{1}{\lambda-1}, \frac{2}{\lambda-1}, \frac{-3}{\lambda-1}\right)$

za $\lambda = -2$ sistem ima ∞ mnogo rješenja $\left(\frac{3t-4}{3}, \frac{3t-5}{3}, t\right)$, $t \in \mathbb{R}$

za $\lambda = 1$ sistem nema rješenja

Riješiti sistem jednačina za razne vrijednosti parametra $\lambda \in \mathbb{R}$:

$$\begin{aligned} 2x_1 - x_2 + 3x_3 - 7x_4 &= 15 \\ 6x_1 - 3x_2 + x_3 - 4x_4 &= 7 \\ 4x_1 - 2x_2 + 14x_3 - 31x_4 &= \lambda \end{aligned}$$

Rj.

Riješimo sistem Kroneker-Kapelijevom metodom:

$$\overline{C} = [C \mid b] = \left[\begin{array}{cccc|c} 2 & -1 & 3 & -7 & 15 \\ 6 & -3 & 1 & -4 & 7 \\ 4 & -2 & 14 & -31 & \lambda \end{array}\right] \underset{III_v - I_v \cdot 2}{\overset{II_v - I_v \cdot 3}{\sim}} \left[\begin{array}{cccc|c} 2 & -1 & 3 & -7 & 15 \\ 0 & 0 & -8 & 17 & -38 \\ 0 & 0 & 8 & -17 & \lambda-30 \end{array}\right]$$

$$\overset{III_v + II_v}{\sim} \left[\begin{array}{cccc|c} 2 & -1 & 3 & -7 & 15 \\ 0 & 0 & -8 & 17 & -38 \\ 0 & 0 & 0 & 0 & \lambda-68 \end{array}\right]$$

1° $\lambda - 68 \neq 0$, $\lambda \neq 68$

$\operatorname{rang} C = 2$, $\operatorname{rang} \overline{C} = 3$, $\operatorname{rang} C < \operatorname{rang} \overline{C}$

Prema Kroneker-Kapelijevoj teoremi sistem nema rješenje

2° $\lambda - 68 = 0$, $\lambda = 68$

$\operatorname{rang} C = \operatorname{rang} \overline{C} = 2 < 4$ (broj nepoznatih)

Prema Kroneker-Kapelijevoj teoremi dvije promjenjive uzimamo proizvoljno, npr. $x_4 = t$, $x_1 = s$

$$\begin{aligned} 2x_1 - x_2 + 3x_3 - 7x_4 &= 15 \\ -8x_3 + 17x_4 &= -38 \end{aligned}$$

$x_4 = t$

$-8x_3 + 17t = -38$

$-8x_3 = -17t - 38$

$x_3 = \frac{17}{8}t + \frac{38}{8} = \frac{17}{8}t + \frac{19}{4}$

$x_1 = s$

$2s - x_2 + 3\left(\frac{17}{8}t + \frac{38}{8}\right) - 7t = 15$

$x_2 = \frac{51t}{8} + \frac{114}{8} + 2s - 7t - 15$

$x_2 = -\frac{5}{8}t - \frac{6}{8} + 2s$

$x_2 = 2s - \frac{5}{8}t - \frac{3}{4}$

Za $\lambda = 68$ rješenje sistema je

$$\left(s,\ 2s - \frac{5}{8}t - \frac{3}{4},\ \frac{17}{8}t + \frac{19}{4},\ t\right),\quad t, s \in \mathbb{R}$$

(#) Riješiti sistem jednačina za razne vrijednosti parametra $\lambda \in \mathbb{R}$:

$$\begin{aligned} 8x_1 + 12x_2 + 7x_3 + \lambda x_4 &= 9 \\ 6x_1 + 9x_2 + 5x_3 + 6x_4 &= 7 \\ 4x_1 + 6x_2 + 3x_3 + 4x_4 &= 5 \\ 2x_1 + 3x_2 + 2x_3 + 2x_4 &= 2. \end{aligned}$$

Rj: Sistem ćemo riješiti Kroneker-Kapelijevom metodom:

$$\overline{B} = [B \mid b] = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 8 & 12 & 7 & \lambda & 9 \\ 6 & 9 & 5 & 6 & 7 \\ 4 & 6 & 3 & 4 & 5 \\ 2 & 3 & 2 & 2 & 2 \end{array}\right] \end{array} \underset{I_v \leftrightarrow IV_v}{\sim} \left[\begin{array}{cccc|c} 2 & 3 & 2 & 2 & 2 \\ 6 & 9 & 5 & 6 & 7 \\ 4 & 6 & 3 & 4 & 5 \\ 8 & 12 & 7 & \lambda & 9 \end{array}\right] \begin{array}{c} II_v - I_v \cdot 3 \\ III_v - I_v \cdot 2 \\ \sim \\ IV_v - I_v \cdot 4 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 2 & 3 & 2 & 2 & 2 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & \lambda-8 & 1 \end{array}\right] \begin{array}{c} III_v - II_v \\ \sim \\ IV_v - II_v \end{array} \left[\begin{array}{cccc|c} 2 & 3 & 2 & 2 & 2 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda-8 & 0 \end{array}\right]$$

1° za $\lambda = 8$ imamo $\operatorname{rang} B = \operatorname{rang} \overline{B} = 2 < 4$ pa prema Kroneker-Kapelijevoj teoremi sistem ima ∞ mnogo rješenja. Dvije promjenjive uzimamo proizvoljno npr. $x_1 = t$, $x_4 = s$

$$\begin{aligned} 2x_1 + 3x_2 + 2x_3 + 2x_4 &= 2 \\ -x_3 + 0x_4 &= 1 \end{aligned}$$

$x_3 = -1$

$2t + 3x_2 - 2 + 2s = 2$

$3x_2 = 4 - 2t - 2s$

$x_2 = \frac{2}{3}(2 - t - s)$

Rješenje sistema (za $\lambda = 8$) je $(t, \frac{2}{3}(2-t-s), -1, s)$ gdje su $s, t \in \mathbb{R}$.

2° za $\lambda \neq 8$ imamo $\operatorname{rang} B = \operatorname{rang} \overline{B} = 3 < 4$ pa prema Kroneker-Kapelijevoj teoremi sistem ima ∞ mnogo rješenja. Jednu promjenjivu uzimamo proizvoljno npr. $x_2 = t$.

$$\begin{aligned} 2x_1 + 3x_2 + 2x_3 + 2x_4 &= 2 \\ -x_3 &= 1 \\ (\lambda - 8)x_4 &= 0 \end{aligned}$$

$x_4 = 0$

$x_3 = -1$

$2x_1 + 3t - 2 = 2$

$2x_1 = 4 - 3t$

$x_1 = 2 - \frac{3}{2}t$

Rješenje sistema (za $\lambda \neq 8$) je $(2 - \frac{3}{2}t, t, -1, 0)$ gdje su $t \in \mathbb{R}$.

(#) Riješiti sistem jednačina za razne vrijednosti parametra $\lambda \in \mathbb{R}$:

$$\begin{aligned} \lambda x_1 - 4x_2 + 9x_3 + 10x_4 &= 11 \\ 2x_1 - x_2 + 3x_3 + 4x_4 &= 5 \\ 4x_1 - 2x_2 + 5x_3 + 6x_4 &= 7 \\ 6x_1 - 3x_2 + 7x_3 + 8x_4 &= 9 \end{aligned}$$

Rj. Sistem ćemo riješiti Kroneker-Kapelijevom metodom:

$$\bar{A} = [A \mid b] = \left[\begin{array}{cccc|c} \lambda & -4 & 9 & 10 & 11 \\ 2 & -1 & 3 & 4 & 5 \\ 4 & -2 & 5 & 6 & 7 \\ 6 & -3 & 7 & 8 & 9 \end{array}\right] \overset{I_v \leftrightarrow IV_v}{\sim} \left[\begin{array}{cccc|c} 6 & -3 & 7 & 8 & 9 \\ 2 & -1 & 3 & 4 & 5 \\ 4 & -2 & 5 & 6 & 7 \\ \lambda & -4 & 9 & 10 & 11 \end{array}\right] \overset{II_v \leftrightarrow I_v}{\sim}$$

$$\sim \left[\begin{array}{cccc|c} 2 & -1 & 3 & 4 & 5 \\ 6 & -3 & 7 & 8 & 9 \\ 4 & -2 & 5 & 6 & 7 \\ \lambda & -4 & 9 & 10 & 11 \end{array}\right] \overset{I_k \leftrightarrow IV_k}{\sim} \begin{array}{c} \begin{array}{cccc} x_4 & x_2 & x_3 & x_1 \end{array} \\ \left[\begin{array}{cccc|c} 4 & -1 & 3 & 2 & 5 \\ 8 & -3 & 7 & 6 & 9 \\ 6 & -2 & 5 & 4 & 7 \\ 10 & -4 & 9 & \lambda & 11 \end{array}\right] \end{array} \overset{I_k \leftrightarrow II_k}{\sim} \begin{array}{c} \begin{array}{cccc} x_2 & x_4 & x_3 & x_1 \end{array} \\ \left[\begin{array}{cccc|c} -1 & 4 & 3 & 2 & 5 \\ -3 & 8 & 7 & 6 & 9 \\ -2 & 6 & 5 & 4 & 7 \\ -4 & 10 & 9 & \lambda & 11 \end{array}\right] \end{array}$$

$$\underset{IV_v - I_v \cdot 4}{\overset{\substack{II_v - I_v \cdot 3 \\ III_v - I_v \cdot 2}}{\sim}} \left[\begin{array}{cccc|c} -1 & 4 & 3 & 2 & 5 \\ 0 & -4 & -2 & 0 & -6 \\ 0 & -2 & -1 & 0 & -3 \\ 0 & -6 & -3 & \lambda-8 & -9 \end{array}\right] \overset{II_k \leftrightarrow IV_k}{\sim} \begin{array}{c} \begin{array}{cccc} x_2 & x_1 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} -1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -2 & -4 & -6 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & \lambda-8 & -3 & -6 & -9 \end{array}\right] \end{array} \overset{III_v \leftrightarrow II_v}{\sim} \left[\begin{array}{cccc|c} -1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & -2 & -4 & -6 \\ 0 & \lambda-8 & -3 & -6 & -9 \end{array}\right]$$

$$\overset{\substack{III_v - II_v \cdot 2 \\ IV_v - II_v \cdot 3}}{\sim} \begin{array}{c} \begin{array}{cccc} x_2 & x_1 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} -1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda-8 & 0 & 0 & 0 \end{array}\right] \end{array}$$

a) Za $\lambda = 8$ imamo $\operatorname{rang} A = \operatorname{rang} \bar{A} = 2 < 4$ pa prema Kroneker-Kapelijevoj teoremi sistem ima ∞ mnogo rješenja.
2. promjenjive uzimamo proizvoljno npr. $x_4 = t$ $x_1 = s$

$$\begin{aligned} -x_3 - 2x_4 &= -3 \\ -x_2 + 2x_1 + 3x_3 + 4x_4 &= 5 \end{aligned}$$

$x_3 = 3 - 2t$

$-x_2 + 2s + 3(3-2t) + 4t = 5$

$x_2 = 2s + 9 - 6t + 4t - 5$

$x_2 = 2s - 2t + 4$

Za $\lambda = 8$ rješenje sistema je $(s, 2s-2t+4, 3-2t, t)$ $t, s \in \mathbb{R}$

b) Za $\lambda \neq 8$ imamo $\operatorname{rang} A = \operatorname{rang} \bar{A} = 3 < 4$ pa prema Kroneker-Kapelijevom teoremu sistem ima ∞ mnogo rješenja.
1. (jednu) promjenjivu uzimamo proizvoljno npr. $x_4 = t$

$$\begin{aligned} (\lambda - 8) x_1 &= 0 \\ -x_3 - 2x_4 &= -3 \\ -x_2 + 2x_1 + 3x_3 + 4x_4 &= 5 \end{aligned}$$

$x_1 = 0$

$x_3 = 3 - 2t$

$-x_2 + 3(3-2t) + 4t = 5$

$x_2 = 9 - 6t + 4t - 5 = -2t + 4$

Za $\lambda \neq 8$ rješenje sistema je $(0, 4-2t, 3-2t, t)$.

Homogeni sistemi linearnih jednačina

Homogeni sistem linearnih jednačina je oblika $A \cdot x = 0$

gdje je
$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} \quad i \quad 0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m\times 1}$$

<u>Teorema:</u> Homogeni sistem ima netrivijalna rješenja akko je $D = 0$ ($\det A = 0$).

1. Riješiti homogeni sistem jednačina
$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \quad (1) \\ 3x_1 + x_2 - x_3 &= 0 \quad (2) \\ 2x_1 + x_2 &= 0 . \end{aligned}$$

Rj. $(1)+(2)$
$$\begin{aligned} 4x_1 + 2x_2 &= 0 \\ 2x_1 + x_2 &= 0 \quad / \cdot 2 \end{aligned}$$
$$\begin{aligned} 4x_1 + 2x_2 &= 0 \\ 4x_1 + 2x_2 &= 0 \end{aligned}$$

$4x_1 + 2x_2 = 0 \quad /:2$

$2x_1 + x_2 = 0$

sistem ima ∞ mnogo rješenja

$x_2 = -2x_1$ $\qquad t - 2t + x_3 = 0$

$x_1 = t$, $x_2 = -2t$, $t \in \mathbb{R}$, $\qquad x_3 = t$

Sistem ima beskonačno mnogo rješenja oblika $(t, -2t, t)$

2. Naći λ tako da sistem
$$\begin{aligned} 3x + y + \lambda z &= 0 \\ 4x - 8y + \lambda z &= 0 \\ 5x - 3y + 3z &= 0 \end{aligned}$$
ima netrivijalna rješenja pa naći rješenja.

Rj.
$$D = \begin{vmatrix} 3 & 1 & \lambda \\ 4 & -8 & \lambda \\ 5 & -3 & 3 \end{vmatrix} \underset{III_v + I_v \cdot 3}{\overset{II_v + I_v \cdot 8}{=\!=}} \begin{vmatrix} 3 & 1 & \lambda \\ 28 & 0 & 9\lambda \\ 14 & 0 & 3\lambda+3 \end{vmatrix} = - \begin{vmatrix} 28 & 9\lambda \\ 14 & 3\lambda+3 \end{vmatrix} = (-14) \cdot 3 \begin{vmatrix} 2 & 3\lambda \\ 1 & \lambda+1 \end{vmatrix} = -42(-\lambda+2)$$

Za $\lambda = 2$ ($D = 0$) u sistemu postoje netrivijalna rješenja.

Sistem sad izgleda:
$$\begin{aligned} 3x + y + 2z &= 0 \quad / \cdot 3 \\ 4x - 8y + 2z &= 0 \quad / \cdot 3 \\ 5x - 3y + 3z &= 0 \quad / \cdot 2 \end{aligned}$$
$$\begin{aligned} 9x + 3y + 6z &= 0 \quad (1) \\ 12x - 24y + 6z &= 0 \quad (2) \\ 10x - 6y + 6z &= 0 \quad (3) \end{aligned}$$

$(3)-(1)$: $x - 9y = 0$

$(2)-(1)$: $3x - 27y = 0 \quad /:3$

$x - 9y = 0$

$x = 9y$, $z = -14y$ $\qquad$ postoji ∞ mnogo rješenja

$(9t, t, -14t)$, $t \in \mathbb{R}$ su rješenja sistema

3. Za koje vrijednosti λ sistem ima netrivijalna rješenja
$$\begin{aligned} \lambda x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 + \lambda x_2 + x_3 + x_4 &= 0 \\ x_1 + x_2 + \lambda x_3 + x_4 &= 0 \\ x_1 + x_2 + x_3 + \lambda x_4 &= 0 . \end{aligned}$$

Rj. za $\lambda = 1$ ili $\lambda = -3$

(Zadaci su skinuti sa stranice: \pf.unze.ba\nabokov
Za uočene greške pisati na **infoarrt@gmail.com**)